Theory of electron retardation by Langmuir probes in anisotropic plasmas

R. Claude Woods and Isaac D. Sudit*
Department of Chemistry, University of Wisconsin, Madison, Wisconsin 53706
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The determination of electron densities and electron velocity distribution functions (EVDF's) from the current-voltage (I-V) characteristics in the electron repelling region is considered for cylindrical, spherical, one-sided planar, and two-sided planar Langmuir probes. Previous treatments of axisymmetric plasmas, in which the EVDF is expressed as a series in Legendre polynomials, are extended and generalized, including full consideration of orbital motion in the arbitrary sheath thickness case for cylindrical probes. An alternative formulation focusing on the first derivative of the I-V data, which is normally more noise immune than the usually used second derivative, is given for one-sided planar probes. A concept of an isotropic EVDF that would give the same probe current as the actual anisotropic one is defined for various probe geometries and used to clarify the physical meaning of parameters extracted from measurements with a single probe orientation. The theory is extended to a completely anisotropic plasma using an expansion of the EVDF in a series of spherical harmonic functions. The geometrical relationships between the various coordinate systems are expressed in terms of the group multiplication rule for the irreducible representations of the three-dimensional rotation group. A method for extracting the complete three-variable EVDF from probe I-V data at a sufficient number of probe orientations is given. The necessary Volterra integral equations are shown to be no more difficult than those arising in the axisymmetric case. Finally, it is shown that the original method of Langmuir or Druyvesteyn for finding electron densities by integrating the second derivative of the I-V characteristic is much more robust towards anisotropy of the plasma than previously realized. Specifically, the usual method, applied exactly as if the plasma were indeed isotropic, should with a single arbitrary orientation of a cylindrical or two-sided planar probe (or with a spherical probe) give the exact electron density, even in a completely anisotropic plasma, and this result is shown to be independent of the ratio of sheath radius to probe radius for cylindrical or spherical probes.

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I. INTRODUCTION

Although the original development of the theory of electron collection by Langmuir probes assumed that the velocity distribution of electrons (and of ions) was isotropic at the spatial point being probed, in many instances of practical interest there exists a moderate or even large degree of anisotropy in the electron velocity space. The ubiquitous occurrence (in low temperature plasmas) of elastic collisions with heavy gas molecules, which tend to randomize the direction of electron motion in just one or two collisions at low energy, provides a very strong tendency towards isotropy, and it is this which has made the original simplifying assumption so useful and widely applied. The method of obtaining the electron energy distribution function (EEDF) in an isotropic plasma from the second derivative of the probe characteristic in the electron repelling region, and in turn of finding the electron density from integrating the EEDF over all energies, has become a standard tool of experimental plasma physics, since it was proposed by Mott-Smith and Langmuir [1] and by Druyvesteyn [2]. In important regions of plasmas near electrodes, or double layers, or electron beam sources, etc., there may, however, be significant degrees of plasma anisotropy, and it is important to measure these anisotropic electron velocity distribution functions (EVDF's) accurately in order to characterize the operation of plasma sources, to understand the flow of electrons and of energy in the plasma, and to check numerical modeling of plasma electron dynamics. Langmuir probes can still do this, as long as data are acquired at several orientations of the probe. As discussed in the next paragraph several papers in the last decade have demonstrated this clearly in the case of an axisymmetric plasma. The present paper extends that treatment in several ways and also provides the basis for obtaining the EVDF in the case where the latter has no symmetry. Many important plasma generation configurations are either not cylindrically symmetrical at all or have their approximate symmetry broken by side ports or other irregularities. Even when a plasma device has cylindrical symmetry, the axisymmetric EVDF assumption is only rigorously valid at on-axis points.

The essentials of the theory for electron collection in axisymmetric plasmas were given by Fedorov [3], as will be described in more detail below. His treatment included one- and two-sided planar probes, cylindrical probes, and spherical probes but used a thin sheath approximation throughout. Mezentsev *et al.* [4] applied the

*Present address: Department of Electrical Engineering, University of California–Los Angeles, 405 Hilgard Avenue, Los Angeles, CA 90024-1594.

theory by using two orientations of a cylindrical probe in the cathode region of a helium arc to find the first two even-order coefficients ($\ell = 0$ and 2) in the EVDF expansion. Mezentsev and Mustafaev [5] extended this work to find the first odd term ($\ell = 1$) by using relationships derived from the Boltzmann equation. Mezentsev *et al.* [6] used three orientations of a one-sided planar probe to find three coefficient functions ($\ell = 0 - 2$) and used the Boltzmann equation relationships to find electron collision cross sections from these. Additional papers by Mezentsev and co-workers [7–9] describe related further experimental work. Fedorov and Mezentsev [10] first gave a general expression for the resolvents in the Volterra integral equations (see below) and briefly discussed the issue of singularities in these equations. Lapshin and Mustafaev [11] discussed numerical methods and error analysis associated with the implementation of Fedorov's method. Klagge and Lunk [12] compared results for a one-sided planar probe using three probe orientations and using five orientations, and for each of these cases they gave convenient explicit forms of the necessary equations. Klagge [13] used three orientations of a one-sided planar probe to measure anisotropy in a 27 MHz rf discharge. Mal'kov [14] has proposed using a method, slightly different from Fedorov's original approach, where instead of using a system of linear equations one integrates current or its derivatives over probe orientation angle with a Legendre polynomial weight. Similar possibilities were also mentioned in Refs. [10,11]. Reference [14] also described the possibility for planar probes of focusing on first derivatives of the current rather than second, as does the present work. Kalinin and Mal'kov [15] further discuss the method of Ref. [14] and also apply it experimentally. Mal'kov [16] considers the collection of electrons by a cylindrical probe without using a thin sheath approximation and obtains explicit results for the coefficients of $P_0(\cos\theta)$ and $P_2(\cos\theta)$. In Ref. [17] numerical modeling is used to examine the accuracy obtainable in the application of Mal'kov's approach [14] to determining the EVDF in the presence of varying degrees of anisotropy.

In the next section (II) of this paper we discuss general concepts of electron collection in the retardation region for planar and cylindrical probes in a plasma with no assumed symmetry. For cylindrical probes and for two-sided planar probes it is proven that the exact density is obtained from Druyvesteyn's usual method [2] at any single orientation of the probe. In the process certain isotropic distribution functions are defined (by the requirement that they produce the same probe current as the actual anisotropic EVDF), and these are also used to illuminate the meaning of the effective temperatures that are obtained from the Druyvesteyn method for each probe type. In Sec. III the axisymmetric case is described and treated by using the same Legendre polynomial expansion of the EVDF that Fedorov did [3]. For planar probes the use of the first derivative of the probe characteristic is considered as an alternative to Fedorov's second derivative based approach. The Volterra integral equation solutions, which are also needed for the nonaxisymmetric case, are discussed in this section. For the cylindrical probe the derivation makes no assumption of a thin sheath, and the final result, though stated in a somewhat different form from Fedorov's, is shown to be equivalent to it. In Sec. IV a form of the theory appropriate when there is no axis of symmetry is developed. Here the EVDF is expanded into a series of spherical harmonic functions, and it is shown that by solving a set of linear equations for one-sided planar probes involving first or second derivatives of probe characteristics at a sufficient set of probe orientations, followed by numerical inversion of the resulting Volterra integral equations, the full EVDF can be obtained. A version of the theory involving only real functions instead of spherical harmonics is also provided. For two-sided planar probes and cylindrical probes it again follows that the density is correctly obtained even from data at a single probe orientation. The spherical probe case is also treated using the spherical harmonic expansion of EVDF. In Sec. V there is a brief discussion of the results of the previous sections and of ways of applying them.

Certain simplifying assumptions are made throughout this paper but are only mentioned in this paragraph. It is assumed that all electrons incident on the probe are collected, that there is no secondary electron emission, and that there are no electron collisions in the sheath. Edge effects at the end of a cylindrical probe or around the perimeter of planar probes are neglected, and the surface of the sheath in front of a planar probe is taken to be perfectly planar too. Use of a guard ring arrangement [18] is suggested as a way of making the latter assumption more realistic. Perturbations of the plasma by either the probe or the probe holder are neglected. We assume that the length scale over which the EVDF changes appreciably is long compared to the physical dimensions of the probe. Of course actual measurements are subject to experimental errors and to noise. Thus in the following sections, when we say a certain result is exact, we mean exact within the framework of all these assumptions, or in other words that it is just as exact as the corresponding result for an isotropic EVDF. Everywhere in this paper I is taken to mean the electron current, not the total current. If the positive ion contribution to the total current or its derivatives cannot be neglected in comparison to that of the electrons in the circumstances of any given experiment, then corrections for it must be made to the observed total probe current before applying the equations of this paper. Our sign convention for I is the usual one, where $I > 0$ corresponds to positive current flowing from the probe into the plasma, i.e., to collected electrons.

II. GENERAL RESULTS FOR THE UNEXPANDED EVDF

A. One-sided planar probe

We begin with the electron velocity distribution function expressed in rectangular coordinates in velocity space and normalized to the electron density:

$$\int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \; f(v_x, v_y, v_z) = n_e. \quad (1)$$

Throughout, f, expressed in whatever variables, will be taken to represent the same numerical value at a given point in velocity space. Consider a one-sided planar probe with the coordinate system oriented so that the positive v_z axis in the velocity space at the sheath edge points inward directly normal to the probe surface. We let V_p represent the probe bias potential and V_s the sheath-edge potential (the plasma potential), which we take as the zero of potential ($V_p - V_s = V_p$). For the retardation region treated in this paper $V_p < 0$. The collected probe current is

$$I(V_p) = A_p e \int_{\sqrt{\frac{-2eV_p}{m}}}^{\infty} dv_z \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \; v_z \times f(v_x, v_y, v_z), \quad (2)$$

where A_p is the area of the probe (taken as equal to the sheath area). Now define the z projected EVDF,

$$H(v_z) = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \; f(v_x, v_y, v_z), \quad (3)$$

and obtain

$$I(V_p) = A_p e \int_{\sqrt{\frac{-2eV_p}{m}}}^{\infty} v_z H(v_z) dv_z. \quad (4)$$

For the first derivative of the probe trace it follows that

$$\frac{dI}{dV_p} = A_p \frac{e^2}{m} H\left(\sqrt{\frac{-2eV_p}{m}}\right), \quad (5)$$

so that $H(v_z)$ for $v_z > 0$ can be obtained directly from the first derivative. On the other hand, the one-sided repelling probe in this single orientation is blind to f or H in the other half of velocity space ($v_z < 0$). We further define the function $\tilde{f}_{z1}(v)$ to mean the isotropic EVDF that yields the same $H(v_z)$ for $v_z > 0$ that the actual f does, i.e., so that

$$\int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \tilde{f}_{z1}\left(\sqrt{v_x^2 + v_y^2 + v_z^2}\right) = H(v_z) \quad \text{for } v_z > 0. \quad (6)$$

In cylindrical coordinates $v_\perp, \phi, v_z$ with v defined by $v^2 = v_\perp^2 + v_z^2$ we can write

$$H(v_z) = \int_0^{2\pi} d\phi \int_0^{\infty} v_\perp f(v_\perp, \phi, v_z) dv_\perp = \int_0^{2\pi} d\phi \int_{|v_z|}^{\infty} v f(v, \phi, v_z)\, dv \quad (7)$$

and for $v_z > 0$

$$H(v_z) = 2\pi \int_{v_z}^{\infty} v \tilde{f}_{z1}(v) dv. \quad (8)$$

Substituting the latter expression into Eq. (5) and differentiating again gives

$$\frac{dI}{dV_p} = 2\pi A_p \frac{e^2}{m} \int_{\sqrt{\frac{-2eV_p}{m}}}^{\infty} v \tilde{f}_{z1}(v)\, dv$$

and

$$\frac{d^2 I}{dV_p^2} = 2\pi A_p \frac{e^3}{m^2} \tilde{f}_{z1}\left(\sqrt{\frac{-2eV_p}{m}}\right). \quad (9)$$

Thus $\tilde{f}_{z1}$, if it exists, will be what we obtain by applying the usual Druyvesteyn second derivative method as if the plasma were isotropic.

If we change variable to $v = \sqrt{-2eV_p/m}$ (the probe bias expressed in velocity units) we can convert the last part of Eq. (9) to the form

$$\tilde{f}_{z1}(v) = \frac{1}{2\pi A_p e} \frac{1}{v} \frac{d}{dv} \frac{1}{v} \frac{d}{dv} I, \quad (10)$$

but using Eq. (4) this becomes

$$\tilde{f}_{z1}(v) = \frac{1}{2\pi} \frac{1}{v} \frac{d}{dv} \frac{1}{v} \frac{d}{dv} \int_{v}^{\infty} v_z H(v_z) dv_z. \quad (11)$$

Equation (11) shows by construction that $\tilde{f}_{z1}$ always exists, because $H(v_z)$ always exists, and substituting Eq. (11) into Eq. (8) immediately leads to self-consistency. If we now turn the probe around 180°, we can obtain $H(v_z)$ for $v_z < 0$ and define an analogous function $\tilde{f}_{z2}(v)$. We could also select other orientations and define $H(v_x), \tilde{f}_{x1}$, etc., in the obvious way. For the electron density and the average value of a property depending on v_z we have

$$n_e = \int_{-\infty}^{\infty} H(v_z) dv_z$$

and

$$\langle T(v_z)\rangle = \frac{1}{n_e} \int_{-\infty}^{\infty} H(v_z) T(v_z) dv_z. \quad (12)$$

In the interesting symmetrical case, where H is an even function, e.g., a plasma symmetry axis lying in the plane of the probe, and $T(v_z)$ is an even property, we have $\tilde{f}_{z1} = \tilde{f}_{z2} \equiv \tilde{f}_z$, and

$$\langle T(v_z)\rangle = \frac{4\pi}{n_e} \int_0^{\infty} T(v_z) \int_{v_z}^{\infty} v \tilde{f}_z(v)\, dv dv_z = \frac{4\pi}{n_e} \int_0^{\infty} \left[\int_0^{+v} T(v_z) dv_z\right] \tilde{f}_z(v)\, v dv. \quad (13)$$

For the z contribution to the kinetic energy, $mv_z^2/2$, we obtain as expected from this formula one-third of the isotropic result:

$$\left\langle \frac{mv_z^2}{2} \right\rangle = \frac{1}{3} \frac{4\pi}{n_e} \int_0^{\infty} \frac{mv^2}{2} \tilde{f}_z(v)\, v^2 dv. \quad (14)$$

B. Two-sided planar probe

Letting A_p equal the combined area of both sides of the probe and combining the contributions of two one-sided probes gives

$$I = \frac{A_p e}{2} \int_{\sqrt{\frac{-2eV_p}{m}}}^{\infty} v_z H(v_z) dv_z - \frac{A_p e}{2} \int_{-\infty}^{-\sqrt{\frac{-2eV_p}{m}}} v_z H(v_z) dv_z$$

and

$$\frac{dI}{dV_p} = \frac{A_p e^2}{m} \frac{1}{2} \left[H\left(\sqrt{\frac{-2eV_p}{m}}\right) + H\left(-\sqrt{\frac{-2eV_p}{m}}\right) \right], \tag{15}$$

which shows that only $[H(v_z) + H(-v_z)]/2$ can be found from two-sided planar probe data, not $H(v_z)$ itself. Similarly we can write analogously to Eq. (11)

$$\tilde{f}_{z2}(v) = -\frac{1}{2\pi} \frac{1}{v} \frac{d}{dv} \frac{1}{v} \frac{d}{dv} \int_{-\infty}^{-v} v_z H(v_z) dv_z \tag{16}$$

and obtain for the second derivative

$$\frac{d^2 I}{dV_p^2} = 2\pi A_p \frac{e^3}{m^2} \frac{1}{2} \left[\tilde{f}_{z1}\left(\sqrt{\frac{-2eV_p}{m}}\right) + \tilde{f}_{z2}\left(\sqrt{\frac{-2eV_p}{m}}\right) \right]. \tag{17}$$

Thus application of the usual Druyvesteyn method determines the average function $(\tilde{f}_{z1} + \tilde{f}_{z2})/2 = \tilde{f}_{z,avg}$ in this case. If we proceeded as if the plasma were isotropic, we would integrate this average function over the entire velocity space to find the electron density:

$$n_e^{calc} = 4\pi \int_0^{\infty} \tilde{f}_{z,avg}(v) v^2 dv. \tag{18}$$

If we substitute Eqs. (11) and (16) and integrate by parts, this reduces to

$$n_e^{calc} = \int_{-\infty}^{\infty} H(v_z) dv_z = n_e^{true}, \tag{19}$$

which proves the following statement. If the electron density in an arbitrarily anisotropic plasma is computed from the integration of the second derivative of the I-V curve of a two-sided planar probe, according to the usual Druyvesteyn prescription for the isotropic case, the exact result will be obtained for any orientation of the probe. Only for even properties $T(v_z)$ can the average be deduced from two-sided probe data, but in this case an exact calculation is possible:

$$\langle T(v_z) \rangle = \frac{2}{n_e} \int_0^{\infty} \frac{H(v_z) + H(-v_z)}{2} T(v_z) dv_z, \tag{20}$$

which leads to analogs of Eqs. (13) and (14) with $\tilde{f}_z$ replaced by $\tilde{f}_{z,avg}$.

C. Cylindrical probe

In this section we consider a cylindrical probe in a completely anisotropic plasma, and since there is no preferred direction we lose no generality by assuming that the probe axis is the polar axis (z) for expressing the EVDF in cylindrical coordinates in velocity space $(v_\perp, \bar{\phi}, v_z)$, where $\bar{\phi}$ is the azimuthal angle between the electron velocity vector and an x axis fixed in the plasma containing vessel. We define a projection of the EVDF (actually an average over angle) onto the perpendicular velocity by

$$G(v_\perp) = \frac{1}{2\pi} \int_0^{2\pi} d\bar{\phi} \int_{-\infty}^{\infty} f(v_\perp, \bar{\phi}, v_z) dv_z. \tag{21}$$

In the special case of an axisymmetric plasma with the probe aligned to the symmetry axis this would reduce to

$$G(v_\perp) = \int_{-\infty}^{\infty} f(v_\perp, v_z) dv_z \tag{22}$$

(axisymmetric, aligned). We let r_p represent the radius of the probe, L its length, and r_s the arbitrary radius of the sheath. The expression for the collected probe current involves three velocity space integrations and two more over the surface of the probe, of which one is trivial, multiplication by the length of the probe. Let $\tilde{\phi}$ be the azimuthal angle from the above mentioned fixed x axis to a small surface element on the probe, and let ϕ be the cylindrical angle from the outward normal to that surface element to the electron velocity vector, so that $\bar{\phi} = \tilde{\phi} + \phi$ (Fig. 1). Following Langmuir's ideas of orbital motion, in which conservation of total energy and of the z component of angular momentum are used to ascertain which electrons starting at the sheath surface reach the probe surface to be collected, we obtain the following limits:

$$v_\perp > (-2eV_p/m)^{\frac{1}{2}}$$

and

$$\pi - \phi^* \leq \phi \leq \pi + \phi^*, \tag{23}$$

where

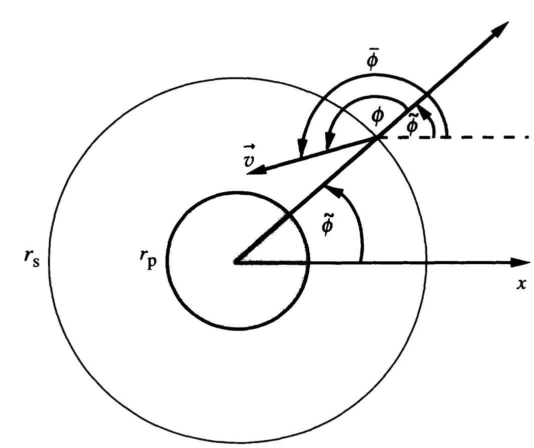

FIG. 1. The azimuthal angles are seen looking down the axis of the cylindrical probe, when the latter is also taken as the polar axis of the vessel-fixed cylindrical coordinate system for expressing the EVDF. The velocity vector is not generally in the plane of the paper; the angles shown are those to its projection in that plane.

$$\phi^* = \sin^{-1}\frac{r_p}{r_s}\sqrt{1+\frac{2eV_p}{mv_\perp^2}}.$$

The collected current can then be written as

$$I = -er_s L\int_0^{2\pi} d\tilde{\phi}\int_{\sqrt{\frac{-2eV_p}{m}}}^{\infty} v_\perp dv_\perp \int_{\pi-\phi^*}^{\pi+\phi^*} d\phi \int_{-\infty}^{\infty} dv_z f(v_\perp,\tilde{\phi},v_z)v_\perp\cos\phi. \tag{24}$$

The $\tilde{\phi}$ integral should be done first, and since it only depends on f, which is periodic in angle, the result is independent of ϕ:

$$\int_0^{2\pi} d\tilde{\phi} f(v_\perp,\phi+\tilde{\phi},v_z) = \int_0^{2\pi} d\tilde{\phi} f(v_\perp,\tilde{\phi},v_z). \tag{25}$$

The ϕ integral no longer depends on the EVDF and can be done explicitly using Eq. (23), after which the sheath radius r_s cancels out. Using the definition in Eq. (21) with a different dummy variable, and remembering that $A_p = 2\pi r_p L$ we come to

$$I = A_p e 2\sqrt{\frac{2}{m}}\int_{\sqrt{\frac{-2eV_p}{m}}}^{\infty} G(v_\perp)\sqrt{\frac{mv_\perp^2}{2}+eV_p}\; v_\perp dv_\perp. \tag{26}$$

This equation shows that the probe current only depends on the distribution of perpendicular velocities, and conversely that only $G(v_\perp)$ and quantities directly dependent upon it can be determined from the retarding Langmuir probe data. For the electron density and for the average of a property $S(v_\perp)$ we have

$$n_e = \int_{\substack{velocity\\ space}} f = 2\pi\int_0^\infty v_\perp G(v_\perp)dv_\perp$$

and

$$\langle S(v_\perp)\rangle = \frac{2\pi}{n_e}\int_0^\infty v_\perp G(v_\perp)S(v_\perp)dv_\perp. \tag{27}$$

Following along lines similar to the above treatment of planar probes we desire a function $\bar{f}(v)$ that will be an isotropic velocity distribution that yields the same $G(v_\perp)$, and thus the same probe current, as the actual anisotropic EVDF (f). If such a function $\bar{f}$ exists, then it is what the usual Druyvesteyn second derivative method should find (since it is isotropic) and integration of it should produce the exact electron density [because it yields the same $G(v_\perp)$ as f does]. In fact we will define $\bar{f}$ by

$$\bar{f}\left(\sqrt{\frac{-2eV_p}{m}}\right) = \frac{m^2}{2\pi A_p e^3}\frac{d^2 I}{dV_p^2}, \tag{28}$$

where I is given by Eq. (26). Thus for any $G(v_\perp)$ $\bar{f}(v)$ exists by construction. Defining v as we did before Eq. (10) and simplifying Eq. (28) with Eq. (26) substituted into it, we get

$$\bar{f}(v) = \frac{1}{\pi}\frac{1}{v}\frac{d}{dv}\frac{1}{v}\frac{d}{dv}\int_v^\infty G(v_\perp)\sqrt{v_\perp^2 - v^2}\, v_\perp dv_\perp. \tag{29}$$

If we wish to calculate the electron density as we would for any other isotropic EVDF we would use

$$n_e^{calc} = 4\pi\int_0^\infty v^2\bar{f}(v)\,dv. \tag{30}$$

Substituting Eq. (29) into Eq. (30), integrating by parts, and differentiating yields

$$n_e^{calc} = 4\int_0^\infty\left[\int_v^\infty v_\perp G(v_\perp)\frac{1}{\sqrt{v_\perp^2 - v^2}}\,dv_\perp\right]dv, \tag{31}$$

and then exchanging the order of integration and doing the v integral explicitly gives

$$n_e^{calc} = 2\pi\int_0^\infty v_\perp G(v_\perp)dv_\perp = n_e^{true}, \tag{32}$$

according to Eq. (27). This has proven the following statement. In an arbitrarily anisotropic plasma with a retarding cylindrical probe at any orientation the usual method of integrating the second derivative of the probe current, applied just as if the plasma were isotropic, will give the exact electron density. Since $\bar{f}$ is isotropic [$\bar{f} = \bar{f}(\sqrt{v_\perp^2+v_z^2})$], we can use Eq. (22) to calculate $G(v_\perp)$, and with the variable change $v = \sqrt{v_\perp^2+v_z^2}$ it becomes

$$G(v_\perp) = 2\int_{v_\perp}^\infty \bar{f}(v)\frac{v\,dv}{\sqrt{v^2 - v_\perp^2}}. \tag{33}$$

Substituting Eq. (29) (with dummy variable $\bar{v}_\perp$) into this yields an identity in G, which after one differentiation by v and integration by parts in $\bar{v}_\perp$ becomes

$$G(v_\perp) = \frac{2}{\pi}\int_{v_\perp}^\infty \frac{1}{\sqrt{v^2 - v_\perp^2}}\frac{d}{dv} \times\int_v^\infty G'(\bar{v}_\perp)\sqrt{\bar{v}_\perp^2 - v^2}\,d\bar{v}_\perp\,dv. \tag{34}$$

After a further differentiation by v, exchange of the order of integration, and use of the integral

$$\int_{v_\perp}^{\bar{v}_\perp}\frac{v\,dv}{\sqrt{v^2 - v_\perp^2}\sqrt{\bar{v}_\perp^2 - v^2}} = \frac{\pi}{2}, \tag{35}$$

followed by use of the fundamental theorem of calculus, the right hand side reduces to $G(v_\perp)$, proving the identity. This confirms that $\bar{f}$ does indeed lead to the same $G(v_\perp)$ as f does.

A property average $\langle S(v_\perp)\rangle$ can also be expressed in terms of the function $\bar{f}$ using Eq. (33) inserted into Eq. (27), which after changing the order of integration becomes

$$\langle S(v_\perp)\rangle = \frac{4\pi}{n_e}\int_0^\infty v\bar{f}(v)\,dv\int_0^v \frac{S(v_\perp)}{\sqrt{v^2-v_\perp^2}}\,v_\perp\,dv_\perp. \tag{36}$$

In the case of the transverse kinetic energy $(m v_\perp^2/2)$ this reduces after some calculation to

$$\left\langle\frac{m v_\perp^2}{2}\right\rangle = \frac{2}{3}\frac{4\pi}{n_e}\int_0^\infty \frac{m v^2}{2}\bar{f}(v)v^2dv \tag{37}$$

or $\frac{2}{3}$ of the isotropic result. For a Maxwell-Boltzmann distribution this is of course $\frac{2}{3}\ (\frac{3}{2}kT_e) = kT_e$. Thus the effective electron temperature calculated from the cylindrical probe characteristic, as if the plasma were isotropic, is actually in an anisotropic plasma the effective transverse temperature

$$kT_{e\perp\ effective} = \left\langle\frac{m v_\perp^2}{2}\right\rangle. \tag{38}$$

For the mean transverse speed we would obtain in the same way

$$\langle v_\perp\rangle = \frac{\pi}{4}\frac{4\pi}{n_e}\int_0^\infty v\bar{f}(v)\,v^2dv \tag{39}$$

or $\pi/4$ times the mean speed in the isotropic case. For a Maxwellian EVDF this reduces to $\langle v_\perp\rangle = (\pi kT_e/2m)^{1/2}$.

In the case of a planar probe the first derivative of the probe current gave the projected EVDF $H(v_z)$ directly [Eq. (5)], but in the cylindrical probe case the first derivative is only expressible as an integral over $G(v_\perp)$,

$$\frac{dI}{dV_p} = 2A_p\frac{e^2}{m}\sqrt{\frac{2}{m}}\int_v^\infty G(v_\perp)\frac{v_\perp dv_\perp}{\sqrt{v_\perp^2-v^2}}. \tag{40}$$

To obtain $G(v_\perp)$ from actual probe data one must differentiate again to get $\bar{f}$ [Eq. (28)] and then integrate according to Eq. (33).

Finally we consider the relationships between the cylindrical probe results and the one-sided planar probe results in the important special case of an axisymmetric plasma, whose symmetry axis (z) is coincident with the cylindrical probe axis and lies in the plane of the planar probe. Then we have $H(v_x) = H(-v_x)$ by symmetry (with the plane of the probe being the yz plane) and $\tilde{f}_{x1} = \tilde{f}_{x2} = \tilde{f}_x$, and intuitively we expect that $\tilde{f}_x = \bar{f}$, since both the probes are sensitive to the same transverse velocity components in this case. We can show that this is indeed the case [remembering $f = f(v_\perp, v_z)$] starting from

$$H(v_x) = \int_{-\infty}^\infty dv_y\int_{-\infty}^\infty dv_z f = \int_{-\infty}^\infty G(v_\perp)dv_y, \tag{41}$$

where we have used Eq. (22). Now we use Eq. (33) and the evenness of $G(v_\perp)$ with respect to v_y $\left(v_\perp^2 = v_x^2+v_y^2,\ dv_y = v_\perp dv_\perp/\sqrt{v_\perp^2-v_x^2}\right)$ to obtain

$$H(v_x) = 4\int_0^\infty\left[\int_{v_\perp}^\infty \frac{\bar{f}(v)\,v dv}{\sqrt{v^2-v_\perp^2}}\right]dv_y. \tag{42}$$

After changing variables from v_y to $v_\perp$, exchanging the order of integration, and using the definite integral in Eq. (35) again, we find that

$$H(v_x) = 2\pi\int_{v_x}^\infty v\bar{f}(v)\,dv. \tag{43}$$

We also have the x analog of Eq. (8), which looks exactly the same except that it contains $\tilde{f}_x$ instead of $\bar{f}$, and differentiating both of these with respect to v_x immediately proves $\bar{f} = \tilde{f}_x$.

III. EXPANSIONS OF THE EVDF IN THE AXISYMMETRIC CASE

A. One-sided planar probe

Fedorov's treatment [3] of the one-sided planar probe begins by converting Eq. (2) to spherical coordinates in velocity space (v, θ_1, ϕ_1) with the polar axis along the inward normal to the probe surface, and further converting the speed to an energy variable $\epsilon = mv^2/2$ with the probe bias represented as $\epsilon^* = -eV_p$. For the probe current this gives

$$I = \frac{2A_pe}{m^2}\int_{\epsilon^*}^\infty \epsilon d\epsilon\int_0^{\cos^{-1}\sqrt{\frac{\epsilon^*}{\epsilon}}}\sin\theta_1\cos\theta_1 d\theta_1 \times\int_0^{2\pi}d\phi_1 f(\epsilon,\theta_1,\phi_1), \tag{44}$$

and for the first and second derivative he obtains (in our notation)

$$\frac{dI}{dV_p} = \frac{A_pe^2}{m^2}\int_{\epsilon^*}^\infty d\epsilon\int_0^{2\pi}d\phi_1 f\left(\epsilon,\cos^{-1}\sqrt{\frac{\epsilon^*}{\epsilon}},\phi_1\right) \tag{45}$$

and

$$\frac{d^2I}{dV_p^2} = \frac{A_pe^3}{m^2}\left[\int_0^{2\pi} f(\epsilon^*,0,\phi_1)\,d\phi_1 - \int_{\epsilon^*}^\infty d\epsilon\int_0^{2\pi}d\phi_1\frac{\partial}{\partial\epsilon^*}f\left(\epsilon,\cos^{-1}\sqrt{\frac{\epsilon^*}{\epsilon}},\phi_1\right)\right]. \tag{46}$$

Next Fedorov expands the EVDF as a series in Legendre polynomials

$$f(\epsilon,\theta) = \sum_{\ell=0}^\infty f_\ell(\epsilon)P_\ell(\cos\theta) \tag{47}$$

and uses the addition theorem

$$P_\ell(\cos\theta) = P_\ell(\cos\theta_1)P_\ell(\cos\theta_0) + 2\sum_{m=1}^{\ell}\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(\cos\theta_1)P_\ell^m(\cos\theta_0) \times \cos m(\phi_1-\phi_0), \tag{48}$$

where θ_0, ϕ_0 are the spherical angles of the probe inward normal in the symmetry axis system of the plasma. When Eqs. (47) and (48) are substituted into Eq. (46) the $m \neq 0$ terms go away in the ϕ_1 integration, and the final result is

$$\frac{d^2I}{dV_p^2}(\theta_0) = 2\pi\frac{A_p e^3}{m^2}\sum_{\ell=0}^{\infty}P_\ell(\cos\theta_0)F_\ell(\epsilon^*), \tag{49}$$

where

$$F_\ell(\epsilon^*) = f_\ell(\epsilon^*) - \int_{\epsilon^*}^{\infty} d\epsilon f_\ell(\epsilon)\frac{\partial}{\partial\epsilon^*}P_\ell\left(\sqrt{\frac{\epsilon^*}{\epsilon}}\right). \tag{50}$$

The basic idea, which has been elaborated and applied in several later papers, is to truncate the expansion at some finite ℓ, to obtain probe data at $\ell+1$ (often evenly spaced) values of θ_0, and to solve the $\ell+1$ equations of type Eq. (49) for F_0 through F_ℓ. Then Eq. (50) for these ℓ's must be solved for $f_\ell(\epsilon^*)$ in terms of $F_\ell(\epsilon^*)$.

Volterra integral equations of the second kind are of the form

$$z(x) = y(x) + \int_0^x K(x,t)y(t)\,dt, \tag{51}$$

where $z(x)$ and the kernel $K(x,t)$ are assumed to be known and y is the unknown function. Volterra [19] has given the solution as

$$y(x) = z(x) + \int_0^x S(x,t)z(t)\,dt, \tag{52}$$

where the resolvent $S(x,t)$ is related to the kernel by Volterra's principle of reciprocity

$$K(x,t) + S(x,t) = \int_x^t S(x,\xi)K(\xi,t)\,d\xi. \tag{53}$$

Fedorov [3] and other authors have pointed out that Eq. (50) is essentially a Volterra equation of the second kind. In fact it can be converted to the exact canonical form [Eq. (51)] by the variable transformation $x = 1/\epsilon^*$ and $t = 1/\epsilon$. For $\ell = 1$ and 2 the resolvent $S(x,t)$ may be found conveniently by Volterra's method of iterative kernels [19]. For $\ell = 3$ and 4 it is easier to substitute a trial solution of the form

$$S(x,t) = ax^pt^q + bx^mt^n \tag{54}$$

into Eq. (53) and solve the resulting equations for the exponents and then the coefficients. Klagge and Lunk [12] have given explicit results for the resolvents up to $\ell = 4$. By inspection of these one can guess that the general solution of Eq. (50) is

$$f_\ell(\epsilon^*) = F_\ell(\epsilon^*) + \int_{\epsilon^*}^{\infty} d\epsilon\, F_\ell(\epsilon)\frac{1}{2\epsilon^*}P_\ell'\left(\sqrt{\frac{\epsilon}{\epsilon^*}}\right). \tag{55}$$

This can be proved in general by substituting Eq. (55) into Eq. (50) and changing the order of integration. After some more manipulations this leads to an equivalent identity

$$\frac{1}{\sqrt{\epsilon^*}}P_\ell'\left(\sqrt{\frac{\epsilon}{\epsilon^*}}\right) - \frac{1}{\sqrt{\epsilon}}P_\ell'\left(\sqrt{\frac{\epsilon^*}{\epsilon}}\right) = \int_{\epsilon^*}^{\epsilon} P_\ell'\left(\sqrt{\frac{\epsilon^*}{s}}\right)P_\ell'\left(\sqrt{\frac{\epsilon}{s}}\right)\frac{ds}{2s^{3/2}} \tag{56}$$

or with the variable changes $a = \epsilon^{*-1/2}, b = \epsilon^{-1/2}, z = s^{-1/2}$ to

$$aP_\ell'\left(\frac{a}{b}\right) - bP_\ell'\left(\frac{b}{a}\right) = \int_b^a P_\ell'\left(\frac{z}{a}\right)P_\ell'\left(\frac{z}{b}\right)dz. \tag{57}$$

This latter identity can be proved by mathematical induction on ℓ, some integrations by parts, and use of Legendre's differential equation and the recurrence formula

$$P_{n+1}'(x) - xP_n(x) = (n+1)P_n(x). \tag{58}$$

Reference [10] has given a general solution for Eq. (50) that is equivalent to Eq. (55) but expressed in a different way.

An alternative approach for the planar probe in an axisymmetric plasma can be based on the projected EVDF $H(v_n)$, where v_n is the inward normal component of electron velocity at an arbitrary orientation of the probe, and the first derivative of the probe current. Intuitively if the projections $H(v_n)$ can be obtained in enough different directions, then the underlying full EVDF f should be derivable from them. Combining Eqs. (5) and (45) gives ($\epsilon = mv^2/2$ and $\epsilon^* = mv_n^2/2$)

$$H(\epsilon^*) = \frac{1}{m}\int_{\epsilon^*}^{\infty} d\epsilon \int_0^{2\pi} d\phi_1 f\left(\epsilon, \cos^{-1}\sqrt{\frac{\epsilon^*}{\epsilon}}, \phi_1\right). \tag{59}$$

Substituting Eqs. (47) and (48) and doing the ϕ_1 integral then gives

$$H(\epsilon^*) = \frac{2\pi}{m}\sum_{\ell=0}^{\infty}P_\ell(\cos\theta_0)\int_{\epsilon^*}^{\infty} d\epsilon f_\ell(\epsilon)P_\ell\left(\sqrt{\frac{\epsilon^*}{\epsilon}}\right), \tag{60}$$

or expressed in the velocity variables (with $v_n = \sqrt{\frac{-2eV_p}{m}} > 0$)

$$H(v_n) = 2\pi\sum_{\ell=0}^{\infty}P_\ell(\cos\theta_0)\int_{v_n}^{\infty} vf_\ell(v)P_\ell\left(\frac{v_n}{v}\right)dv. \tag{61}$$

Then we define ($v_n > 0$)

$$Q_\ell(v_n) = \int_{v_n}^{\infty} vf_\ell(v)P_\ell\left(\frac{v_n}{v}\right)dv$$

so

$$\frac{dI}{dV_p}(\theta_0) = 2\pi A_p \frac{e^2}{m} \sum_{\ell=0}^{\infty} P_\ell(\cos\theta_0)\, Q_\ell(v_n) \tag{62}$$

and solve a truncated set of linear equations [Eq. (62)] involving first derivatives of probe characteristics at $\ell+1$ orientations θ_0 to find numerically $Q_0(v_n)$ through $Q_\ell(v_n)$. If in Eq. (62) we make the variable changes $t = 1/v$ and $x = 1/v_n$ we get

$$Q_\ell(x) = \int_0^x \frac{1}{t^3} P_\ell\left(\frac{t}{x}\right) f_\ell(t)\, dt, \tag{63}$$

which is an example of a Volterra equation of the first kind [19], namely,

$$z(x) = \int_0^x K(s,t) y(t)\, dt. \tag{64}$$

The latter is converted to a Volterra equation of the second kind by Volterra's method of differentiation [19] [requiring $K(x,x) \neq 0$]

$$\frac{z'(x)}{K(x,x)} = y(x) + \int_0^x \left[\frac{\frac{\partial}{\partial x} K(x,t)}{K(x,x)}\right] y(t)\, dt. \tag{65}$$

Applying this procedure to Eq. (63) and solving the resulting equations for $\ell = 0-4$ by the techniques already referred to (and then converting from x and t back to v_n and v) we guess that the general solution to the first part of Eq. (62) is

$$f_\ell(v_n) = -\frac{Q'_\ell(v_n)}{v_n} - \frac{1}{v_n^2}\int_{v_n}^{\infty} P'_\ell\left(\frac{v}{v_n}\right) Q'_\ell(v)\, dv, \tag{66}$$

or integrating by parts

$$f_\ell(v_n) = -\frac{Q'_\ell(v_n)}{v_n} + \frac{\ell(\ell+1)}{2}\frac{Q_\ell(v_n)}{v_n^2} + \frac{1}{v_n^3}\int_{v_n}^{\infty} P''_\ell\left(\frac{v}{v_n}\right) Q_\ell(v)\, dv. \tag{67}$$

Equation (66) can be proven by differentiating the first line of Eq. (62) once and then substituting Eq. (66) into it. After exchanging the order of integration in the double integral term, setting the coefficient of $Q'_\ell(v)$ in the integral equal to zero, and making the variable changes $a = v_n^{-1}, b = v^{-1}$, and $z = s^{-1}$, we come to the identity Eq. (57) again. After finding the first few Q_ℓ's numerically from probe data as described following Eq. (62), we determine the corresponding coefficient functions in the Legendre polynomical expansion of the EVDF by performing the numerical integration and differentiation in Eq. (67). Inspection of the latter shows that to find the EVDF we must eventually use numerical second derivatives of the probe current, if only in the first term. There may still be advantages to this method in terms of noise immunity in that the critical stage of solving the linear equations [Eq. (61)] to separate the contributions of the different orders ℓ (and seeing where the expansion can be safely truncated) is done with the more noise free numerical first derivatives.

Multiplying Eq. (47) (with velocity as a variable rather than ϵ) by $P_\ell(\cos\theta)\sin\theta$, integrating, and using the orthogonality of the Legendre polynomials we obtain

$$f_\ell(v) = \frac{2\ell+1}{2}\int_0^{\pi} P_\ell(\cos\theta) f(v,\theta)\sin(\theta)\, d\theta \tag{68}$$

and in particular

$$f_0(v) = \frac{1}{2}\int_0^{\pi} f(v,\theta)\sin(\theta)\, d\theta. \tag{69}$$

Thus

$$n_e = \int_{\substack{velocity\\ space}} f = 2\pi \int_0^{\infty} v^2\, dv \int_0^{\pi} \sin(\theta) d\theta f(v,\theta) = 4\pi \int_0^{\infty} v^2 f_0(v)\, dv, \tag{70}$$

so that the electron density is completely determined by the isotropic term in the EVDF expansion.

B. Two-sided planar probes

As pointed out by Fedorov [3], for a two-sided probe the two sides correspond in Eq. (49) to θ_0 and $\pi - \theta_0$, and since $P_\ell[\cos(\pi-\theta_0)] = (-1)^\ell P_\ell(\cos\theta_0)$, the odd-order terms cancel out. If A_p is now understood as the total area of both sides of the probe, the second derivative becomes

$$\frac{d^2I}{dV_p^2}(\theta_0) = 2\pi A_p \frac{e^3}{m^2}\sum_{j=0}^{\infty} P_{2j}(\cos\theta_0) F_{2j}(\epsilon^*). \tag{71}$$

Based on our discussion in Sec. II we should expect the integral of the second derivative method to give the true electron density, i.e.,

$$n_e^{calc} = 4\pi\int_0^{\infty} \frac{m^2}{2\pi A_p e^3}\frac{d^2I}{dV_p^2} v_n^2 dv_n = 4\pi \sum_{j=0}^{\infty} P_{2j}(\cos\theta_0)\int_0^{\infty} F_{2j}(\epsilon^*)\, v_n^2 dv_n = \frac{4\pi}{m}\sqrt{\frac{2}{m}}\sum_{j=0}^{\infty} P_{2j}(\cos\theta_0)\int_0^{\infty} F_{2j}(\epsilon^*)\epsilon^{*1/2}\, d\epsilon^* \tag{72}$$

should reduce to the true n_e. Based on Eq. (70) we can see that this will certainly happen if the integral in Eq. (72) vanishes for $j \neq 0$ [remembering that $F_0(\epsilon^*) = f_0(\epsilon^*)$]. Substituting Eq. (50) into the last form of this integral, exchanging the order of integration, and integrating by parts the integral becomes

$$\int_0^{\infty} F_{2j}(\epsilon^*)\epsilon^{*1/2}\, d\epsilon^* = \frac{1}{2}\int_0^{\infty} f_{2j}(\epsilon)\int_0^{\epsilon}\frac{1}{\sqrt{\epsilon^*}} P_{2j}\left(\sqrt{\frac{\epsilon^*}{\epsilon}}\right) d\epsilon^* d\epsilon. \tag{73}$$

Making the variable change $x = \sqrt{\epsilon^*/\epsilon}$ the inner integral becomes

$$\int_0^\epsilon \frac{1}{\sqrt{\epsilon^*}} P_{2j}\left(\sqrt{\frac{\epsilon^*}{\epsilon}}\right) d\epsilon^* = 2\sqrt{\epsilon}\int_0^1 P_{2j}(x)\,dx$$
$$= \sqrt{\epsilon}\int_{-1}^1 P_{2j}(x)P_0(x)\,dx$$
$$= 2\sqrt{\epsilon}\,\delta_{j0}, \quad (74)$$

as desired. The same argument would fail for a one-sided planar probe, because the next to the last step in Eq. (74) depends upon the evenness of $P_{2j}(x)$. Terms involving $F_{2j+1}(\epsilon^*)$ do not give vanishing integrals.

C. Cylindrical probes

In order to obtain results that are known to be valid for any Debye length to probe radius ratio, we consider electrons entering a sheath of arbitrary radius r_s and subject to collisionless orbital motion inside this sheath. At any element of surface area on this sheath we choose a spherical coordinate system in velocity space with a polar axis parallel to the probe axis and with $\phi = 0$ corresponding to velocities pointing outward in a plane containing the probe axis (Fig. 2). Combining conservation of energy and of the z component of angular momentum with the requirement that the electron trajectory reach the probe surface leads to the following limits on the velocity space coordinates:

$$\pi - \phi^* \le \phi \le \pi + \phi^*,$$

where

$$\phi^* = \sin^{-1}\frac{r_p}{r_s}\sqrt{1+\frac{2eV_p}{mv^2\sin^2\theta}},$$

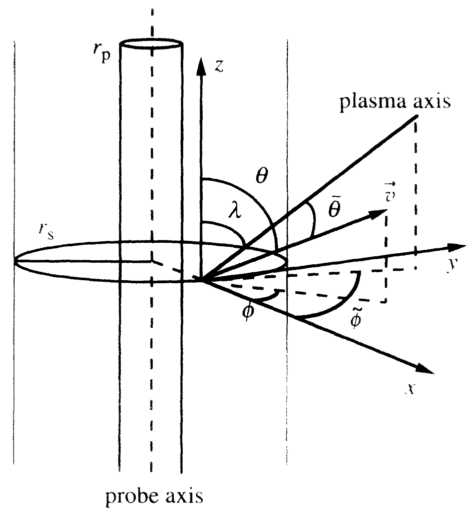

FIG. 2. Orientations of the probe axis, the plasma symmetry axis, and the velocity vector are shown in the case of an axisymmetric plasma.

$$\theta^* \le \theta \le \pi - \theta^*,$$

where

$$\theta^* = \sin^{-1}\sqrt{\frac{-2eV_p}{mv^2}},$$

and

$$v \ge \sqrt{\frac{-2eV_p}{m}}. \quad (75)$$

The expression for the collected current is

$$I = -er_sL\int_0^{2\pi} d\tilde{\phi}\int_{\sqrt{\frac{-2eV_p}{m}}}^{\infty} v^3dv\int_{\theta^*}^{\pi-\theta^*}\sin^2(\theta)d\theta\int_{\pi-\phi^*}^{\pi+\phi^*}\cos(\phi)d\phi f(v,\bar{\theta}). \quad (76)$$

Here θ, ϕ are the polar angles of the electron velocity vector in our spherical coordinate system, $\tilde{\phi}$ is the angular position of this surface element on the sheath boundary, referenced to the plane containing the probe axis and plasma symmetry axis, and λ will be the angle between the two latter axes (Fig. 2). Thus $\lambda, \tilde{\phi}$ will also be the polar angles of the plasma symmetry axis in our coordinate system, and the angle between the plasma symmetry axis and the electron velocity vector ($\bar{\theta}$) will obey an equation of the form of Eq. (48), giving $P_\ell(\cos\bar{\theta})$ in terms of the pairs θ, ϕ and $\lambda, \tilde{\phi}$. We begin by substituting that addition theorem into the Legendre polynomial expansion of $f(v,\bar{\theta})$ and then inserting that into Eq. (76). The $\tilde{\phi}$ integral must be done first, and this eliminates all the $m \neq 0$ terms giving

$$I = -er_sL2\pi\sum_{\ell=0}^{\infty}P_\ell(\cos\lambda)\int_{\sqrt{\frac{-2eV_p}{m}}}^{\infty}v^3dv\int_{\theta^*}^{\pi-\theta^*}\sin^2(\theta)d\theta\int_{\pi-\phi^*}^{\pi+\phi^*}\cos(\phi)d\phi f_\ell(v)P_\ell(\cos\theta). \quad (77)$$

The ϕ integral may now be done explicitly, after which r_s cancels out, and we obtain

$$I = 2A_pe\sum_{\ell=0}^{\infty}P_\ell(\cos\lambda)\int_{\sqrt{\frac{-2eV_p}{m}}}^{\infty}v^3f_\ell(v)\,dv\int_{\theta^*}^{\pi-\theta^*}\sin^2\theta\sqrt{1+\frac{2eV_p}{mv^2\sin^2\theta}}P_\ell(\cos\theta)\,d\theta. \quad (78)$$

It can be seen that the odd ℓ terms give vanishing contributions to this current, because of the oddness of the θ integrand about $\theta = \pi/2$, and we define

$$f_{even}(v,\theta) = \sum_{j=0}^{\infty}P_{2j}(\cos\lambda)P_{2j}(\cos\theta)f_{2j}(v). \quad (79)$$

For the special case of a probe aligned along the plasma axis ($\lambda = 0$) this is just the even part of the EVDF. Now we have

$$I = 4A_p e \int_{\sqrt{\frac{-2eV_p}{m}}}^{\infty} v^3 \int_{\theta^*}^{\pi/2} \sin\theta \sqrt{1 + \frac{2eV_p}{mv^2} - \cos^2\theta}\, f_{even}(v,\theta)\, d\theta. \tag{80}$$

Defining $y = \cos\theta$, $a = \left[1 + 2eV_p/(mv^2)\right]^{1/2}$, and ϵ and ϵ^* as before, this can be written in the form

$$I = \frac{8A_p e}{m^2} \int_{\epsilon^*}^{\infty} \epsilon \int_0^a f_{even}(\epsilon, \cos^{-1} y) \sqrt{a^2 - y^2}\, dy\, d\epsilon, \tag{81}$$

and the first derivative by V_p is

$$\frac{dI}{dV_p} = \frac{4A_p e^2}{m^2} \int_{\epsilon^*}^{\infty} \int_0^a \frac{f_{even}(\epsilon, \cos^{-1} y)}{\sqrt{a^2 - y^2}}\, dy\, d\epsilon. \tag{82}$$

Integrating by parts in the y integral yields

$$\frac{dI}{dV_p} = \frac{4A_p e^2}{m^2} \int_{\epsilon^*}^{\infty} \left\{ \frac{\pi}{2} f_{even}(\epsilon, \cos^{-1} a) - \int_0^a \sin^{-1}\left(\frac{y}{a}\right) \frac{d}{dy} f_{even}(\epsilon, \cos^{-1} y)\, dy \right\} d\epsilon. \tag{83}$$

The second derivative after cancellation of two terms is

$$\frac{d^2 I}{dV_p^2} = 4A_p \frac{e^3}{m^2} \left\{ \frac{\pi}{2} f_{even}\left(\epsilon^*, \frac{\pi}{2}\right) + \int_{\epsilon^*}^{\infty} \frac{1}{2a^2\epsilon} \int_0^a \frac{y}{\sqrt{a^2 - y^2}} \frac{d}{dy} f_{even}(\epsilon, \cos^{-1} y)\, dy \right\} d\epsilon. \tag{84}$$

Reinserting what f_{even} actually is [Eq. (79)] converts this to

$$\frac{d^2 I}{dV_p^2} = 2\pi \frac{A_p e^3}{m^2} \sum_{j=0}^{\infty} P_{2j}(\cos\lambda) \left[P_{2j}(0) f_{2j}(\epsilon^*) + \int_{\epsilon^*}^{\infty} \frac{f_{2j}(\epsilon)}{\pi a^2 \epsilon} \int_0^a P'_{2j}(y) \frac{y\, dy}{\sqrt{a^2 - y^2}}\, d\epsilon \right]. \tag{85}$$

In order to make extracting the EVDF from this a tractable problem it is imperative to evaluate the integral over y, and indeed we have proven the following identity in the Legendre polynomials:

$$\frac{1}{\pi a^2 \epsilon} \int_0^a P'_{2j}(y) \frac{y\, dy}{\sqrt{a^2 - y^2}} = -P_{2j}(0) \frac{d}{d\epsilon^*} P_{2j}\left(\sqrt{\frac{\epsilon^*}{\epsilon}}\right), \tag{86}$$

where $a^2 = 1 - \epsilon^*/\epsilon$. This may be checked explicitly for the first few values of j. To deal with the general case we write

$$P_{2j}(y) = \sum_{i=0}^{j} c_i^j y^{2i},$$

with

$$c_i^j = \frac{(-1)^{i+j}(2i+2j)!}{2^{2j}(2i)!(i+j)!(j-i)!}. \tag{87}$$

We also need

$$\int_0^a \frac{y^{2i}}{\sqrt{a^2 - y^2}}\, dy = \frac{\pi (2i)! a^{2i}}{2(2^i i!)^2} \tag{88}$$

and

$$P_{2j}(0) = (-1)^j \frac{(2j)!}{(2^j j!)^2}. \tag{89}$$

Inserting these into Eq. (86), expanding $a^{2i} = (1 - \epsilon^*/\epsilon)^i$ in a binomial series, exchanging the order of summation, and equating coefficients of $\epsilon^{*i}/\epsilon^{i+1}$ on both sides of the equation, we come to an equivalent identity:

$$\sum_{i=k}^{j-1} \frac{(-1)^i (2i+2j+2)!}{2^{2i+2}(i+1)!k!(i-k)!(j+i+1)!(j-i-1)!} = (-1)^{j-1} \frac{(2j)!\,(k+1)(2j+2k+2)!}{2^{2j}(j!)^2(2k+2)!(j+k+1)!(j-k-1)!}. \tag{90}$$

The meaning of this complicated equation is made more clear by looking at the specific case $j-k = 5$; after getting a common denominator it becomes

$$(2j)(2j-2)(2j-4)(2j-6) - 4(2j)(2j-2)(2j-4)(4j-7) + 6(2j)(2j-2)(4j-7)(4j-5)$$
$$-4(2j)(4j-7)(4j-5)(4j-3) + (4j-7)(4j-5)(4j-3)(4j-1) = (-2j+1)(-2j+3)(-2j+5)(-2j+7). \tag{91}$$

Other values of $j-k$ are analogous with more or less terms and factors [always alternating signs, binomial coefficients, and factors beginning at $(2j), (4j-1)$, and $(-2j+1)$ and incrementing by twos]. This follows from a more general result like (with various number of terms)

$$x(x-a)(x-2a)-3yx(x-a)+3y(y+a)x-y(y+a)(y+2a)=(x-y)(x-y-a)(x-y-2a), \tag{92}$$

which is proved by mathematical induction on the number of terms.

We can now simplify Eq. (85) by using Eq. (86)

$$\frac{d^2I}{dV_p^2}(\lambda)=2\pi\frac{A_pe^3}{m^2}\sum_{j=0}^{\infty}P_{2j}(\cos\lambda)P_{2j}(0)F_{2j}(\epsilon^*), \tag{93}$$

where $F_{2j}(\epsilon^*)$ is the same as in Eq. (50) with $\ell\to 2j$. The inversion of the Volterra integral equation of the second kind proceeds exactly as already discussed in Sec. III A. Again the first several F_{2j} functions are found by solving a system of linear equations [Eq. (93)] with numerical second derivatives at the appropriate number of probe orientations (λ). The coefficient matrix differs from that for a two-sided planar probe only by the presence of the extra factor of $P_{2j}(0)$ [see Eq. (89)].

Fedorov [3] has given an expression for the second derivative derived in the thin sheath approximation $\lambda_D \ll r_p$, which differs from Eq. (93) (besides trivial notation changes) by the substitution

$$P_{2j}(0)P_{2j}(\cos\lambda)=\frac{1}{\pi}\int_0^{\pi}P_{2j}(\sin\theta\sin\lambda)\,d\theta. \tag{94}$$

The identity Eq. (94) can be proven by the same methods used above to prove Eq. (86), to which it can be shown to be equivalent after some manipulation.

The discussion of the electron density in the preceding subsection on two-sided planar probes can be taken over almost unchanged to the cylindrical probe case. When Eq. (93) is substituted into the first line of Eq. (72), all the $\ell\neq 0$ terms again vanish, and the remaining extra factor of $P_0(0)$ is just unity. Again we conclude that the calculation of the electron density in the anisotropic plasma, carried out exactly as if the plasma were isotropic, yields the exact result for a cylindrical probe. Finally we point out that the two isotropic functions $f_0(v)$ and $\bar{f}(v)$, both of which when integrated give the exact electron density [Eqs. (30) and (32) and Eq. (70)], are in general quite different. This can be seen clearly by consideration of a shifted Maxwell-Boltzmann distribution (z is the probe axis)

$$f\propto\exp\left[-\frac{m}{2kT}[v_x^2+v_y^2+(v_z-v_d)^2]\right], \tag{95}$$

where the drift velocity is large compared to the thermal velocity. Our function $\bar{f}$ would be the same as if v_d were 0, whereas f_0 would be shifted to a spherical shell of radius v_d in velocity space.

IV. A PLASMA WITH NO VELOCITY SPACE SYMMETRY

A. One-sided planar probe

In this section we will develop a generalized theory applicable to an EVDF with no particular symmetry. The EVDF in this case can be expanded in a series of spherical harmonic functions

$$f(v,\theta,\phi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}f_{\ell m}(v)Y_{\ell m}(\theta,\phi). \tag{96}$$

Since $f(v,\theta,\phi)$ is always real, the coefficient functions $f_{\ell m}(v)$ must be complex. Consider three Cartesian coordinate systems in velocity space: (I) a system (X,Y,Z) parallel to a similar set in ordinary space fixed in the plasma containment vessel, so that the coordinates of the electron velocity vector are (v_X,v_Y,v_Z), (II) a system (x,y,z) parallel to a configuration space set with the z axis normal to the probe plane (pointing inward) and x,y fixed somehow in the plane of the sheath surface, (v_x,v_y,v_z) being the electron velocity components, and (III) a set (x',y',z') such that z' is along the velocity vector of the electron at the sheath boundary. The orientation of x' and y' is arbitrary, but some definite selection should be imagined at first. There are then three coordinate transformations R, R_0, and R_1 and their corresponding sets of Euler angles that describe rotations from one of these systems to another as follows:

$$\begin{aligned}R(\phi,\theta,\chi)&:\ \text{(I)}\to\text{(III)},\\ R_0(\phi_0,\theta_0,\chi_0)&:\ \text{(I)}\to\text{(II)},\\ R_1(\phi_1,\theta_1,\chi_1)&:\ \text{(II)}\to\text{(III)}.\end{aligned} \tag{97}$$

Thus (θ,ϕ) are the spherical coordinate angles of the velocity vector in the vessel-fixed system, (θ_0,ϕ_0) those of the probe normal vector in the same system, and (θ_1,ϕ_1) those of the velocity vector in the system fixed on the probe. These transformations obey the multiplication law for group elements in the three-dimensional rotation group $R(3)$.

$$R(\phi,\theta,\chi)=R_0(\phi_0,\theta_0,\chi_0)R_1(\phi_1,\theta_1,\chi_1). \tag{98}$$

Note that the first applied element is on the left in the right hand side of Eq. (98), because these are coordinate axes rotations, not the physical system rotations seen commonly in quantum mechanics, where the order of group operation application is from right to left. Using the homomorphism property for the group representations gives

$$D^{(\ell)}_{mm'}(\phi,\theta,\chi) = \sum_{m''=-\ell}^{+\ell} D^{(\ell)}_{mm''}(\phi_0,\theta_0,\chi_0) \times D^{(\ell)}_{m''m'}(\phi_1,\theta_1,\chi_1), \quad (99)$$

where the $D^{(\ell)}$ are the irreducible representation matrices of $R(3)$. Using the relationships

$$D^{(\ell)}_{mm'}(\phi,\theta,\chi) = e^{-im\phi} d^{(\ell)}_{mm'}(\theta) e^{-im'\chi},$$
$$d^{(\ell)}_{mm'}(\theta) = d^{(\ell)*}_{mm'}(\theta), \quad (100)$$
$$D^{(\ell)}_{m0}(\phi,\theta,\chi) = D^{(\ell)}_{m0}(\phi,\theta,0),$$

and

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\, D^{(\ell)^*}_{m0}(\phi,\theta,0),$$

we can specialize Eq. (99) to the case $m' = 0$, take its complex conjugate, and arrive at the equation

$$Y_{\ell m}(\theta,\phi) = \sum_{m''} D^{(\ell)*}_{mm''}(\phi_0,\theta_0,\chi_0) Y_{\ell m''}(\theta_1,\phi_1). \quad (101)$$

For the EVDF this gives the needed expansion

$$f(v,\theta,\phi) = \sum_{\ell m m''} f_{\ell m}(v) D^{(\ell)*}_{mm''}(\phi_0,\theta_0,\chi_0) Y_{\ell m''}(\theta_1 \phi_1), \quad (102)$$

and the insertion of this into Eq. (59) gives

$$H(\epsilon^*) = \frac{1}{m}\int_{\epsilon^*}^{\infty} d\epsilon \int_0^{2\pi} d\phi_1 \sum_{\ell m m''} f_{\ell m}(\epsilon) \times D^{(\ell)*}_{mm''}(\phi_0,\theta_0,\chi_0) Y_{\ell m''}\left(\cos^{-1}\sqrt{\frac{\epsilon^*}{\epsilon}}, \phi_1\right). \quad (103)$$

We can simplify this by doing the ϕ_1 integral first:

$$\int_0^{2\pi} d\phi_1 Y_{\ell m''}\left(\cos^{-1}\sqrt{\frac{\epsilon^*}{\epsilon}}, \phi_1\right) = 2\pi\sqrt{\frac{2\ell+1}{4\pi}} P_\ell\left(\sqrt{\frac{\epsilon^*}{\epsilon}}\right) \delta_{m'',0}. \quad (104)$$

Then we do the sum over m'' and use Eq. (100) again to obtain the general result for the projected EVDF

$$H(\epsilon^*) = \frac{2\pi}{m}\sum_{\ell m} Y_{\ell m}(\theta_0,\phi_0) \int_{\epsilon *}^{\infty} f_{\ell m}(\epsilon) P_\ell\left(\sqrt{\frac{\epsilon^*}{\epsilon}}\right) d\epsilon, \quad (105)$$

which is directly proportional to the first derivative of the one-sided probe I-V data at the orientation defined by the angles (θ_0,ϕ_0) according to Eq. (5). Notice that the arbitrary angle χ_0 disappeared in the last step. Differentiating Eq. (105) once with respect to $V_p = -\epsilon^*/e$ gives for the second derivative

$$\frac{d^2 I}{dV_p^2}(\theta_0,\phi_0) = 2\pi\frac{A_p e^3}{m^2}\sum_{\ell m} Y_{\ell m}(\theta_0,\phi_0) F_{\ell m}(\epsilon^*), \quad (106)$$

where

$$F_{\ell m}(\epsilon^*) = f_{\ell m}(\epsilon^*) - \int_{\epsilon^*}^{\infty} f_{\ell m}(\epsilon)\frac{\partial}{\partial\epsilon^*} P_\ell\left(\sqrt{\frac{\epsilon^*}{\epsilon}}\right) d\epsilon. \quad (107)$$

The determination of the EVDF now proceeds along the same lines as in the axisymmetric case. Some set of (ℓm) are chosen for which the $f_{\ell m}$ functions are to be determined. Second derivative data are obtained at an appropriate number of well chosen orientations (θ_0,ϕ_0). The matrix of coefficients in the system of linear equations [Eq. (106)], which depends on the (complex) spherical harmonic functions, is inverted, and the inverse applied to the column vector of second derivatives to find the selected $F_{\ell m}$ functions. Then the Volterra integral equations of the second kind [Eq. (107)] are solved to find the corresponding coefficient functions in the EVDF [$f_{\ell m}(v)$]. The Volterra integral equations are identical to (have the same kernel and limits as) those arising in the axisymmetric case already treated, and all the discussion of the general solution still applies. Similarly a first derivative oriented method can be based on Eq. (105) by defining (and solving linear equations for)

$$Q_{\ell m}(\epsilon^*) = \frac{1}{m}\int_{\epsilon^*}^{\infty} f_{\ell m}(\epsilon) P_\ell\left(\sqrt{\frac{\epsilon^*}{\epsilon}}\right) d\epsilon$$

or

$$Q_{\ell m}(v_n) = \int_{v_n}^{\infty} v f_{\ell m}(v) P_\ell\left(\frac{v_n}{v}\right) dv \quad (108)$$

and

$$\frac{dI}{dV_p}(\theta_0,\phi_0) = 2\pi A_p \frac{e^2}{m}\sum_{\ell m} Y_{\ell m}(\theta_0,\phi_0)\, Q_{\ell m}(v_n). \quad (109)$$

Equation (108) leads to exactly the same Volterra integral equations of the first kind already discussed. (Again both the kernel and the resolvent are independent of m.) These equations should (and do) reduce to those of the preceding section in the special case of an axisymmetric plasma, in which the terms in Eq. (96) for $m \neq 0$ vanish. The coefficient $f_{\ell m}(v)$ must be normalized slightly differently from $f_\ell(v)$ as follows:

$$f(v,\theta) = \sum_{\ell=0}^{\infty} f_{\ell 0}(v) Y_{\ell 0}(\theta,\phi) = \sum_{\ell} f_{\ell 0}(v)\sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta), \quad (110)$$

so

$$f_\ell(v) = \sqrt{\frac{2\ell+1}{4\pi}} f_{\ell 0}(v),$$
$$F_\ell(v) = \sqrt{\frac{2\ell+1}{4\pi}} F_{\ell 0}(v), \quad (111)$$

and

$$Q_\ell(v_n) = \sqrt{\frac{2\ell+1}{4\pi}}\, Q_{\ell 0}(v_n).$$

In practice it may be convenient to expand the EVDF in a sine/cosine series and avoid all use of complex numbers:

$$f(v,\theta,\phi) = \sum_{\ell=0}^{\infty}\left[f_\ell(\epsilon)P_\ell(\cos\theta) + \sum_{m=1}^{\ell} P_\ell^m(\cos\theta) \times \left\{ g_{\ell m}(\epsilon)\cos m\phi + h_{\ell m}(\epsilon)\sin m\phi \right\}\right]. \tag{112}$$

Because f is a real function and $Y_{\ell m}^*(\theta,\phi) = (-1)^m Y_{\ell,-m}(\theta,\phi)$, the coefficients in the spherical harmonic expansion are constrained by

$$f_{\ell m}^*(\epsilon) = (-1)^m f_{\ell,-m}(\epsilon). \tag{113}$$

Equating the sums of the two terms with fixed ℓ and $|m|$ in the two series establishes the relationships ($m > 0$)

$$g_{\ell m}(\epsilon) = (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\; 2\mathrm{Re}[f_{\ell m}(\epsilon)], \tag{114}$$

$$h_{\ell m}(\epsilon) = -(-1)^m \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}}\; 2\mathrm{Im}[f_{\ell m}(\epsilon)].$$

Clearly the real functions $g_{\ell m}$ and $h_{\ell m}$ and the complex set $f_{\ell m}$ are instantly obtainable from each other. Because the kernel is real, there is no mixing of the real and imaginary parts of $f_{\ell m}$ in the definition of $F_{\ell m}$ [Eq. (107)]. Thus we can define $G_{\ell m}$ from $g_{\ell m}$ and $H_{\ell m}$ from $h_{\ell m}$ in exact analogy to the latter equation. For the second derivative of the I-V curve we get

$$\frac{d^2I}{dV_p^2} = 2\pi\frac{A_p e^3}{m^2}\sum_{\ell=0}^{\infty}\left[F_\ell(\epsilon^*)P_\ell(\cos\theta_0) + \sum_{m=1}^{\ell} P_\ell^m(\cos\theta_0)\left\{G_{\ell m}(\epsilon^*)\cos m\phi_0 + H_{\ell m}(\epsilon^*)\sin m\phi_0\right\}\right]. \tag{115}$$

The Volterra integral equations for $g_{\ell m}$ and $h_{\ell m}$ and their resolvents are the same as those for f_ℓ given in the preceding section. If Eq. (115) is taken as the set of linear equations to be solved, then the coefficient matrix and its inverse are real. A completely real formulation can also be made of the first derivative approach based on the $H(v_n)$. Instead of $Q_{\ell m}(v_n)$ we would define

$$R_{\ell m}(v_n) = \int_{v_n}^{\infty} v\, g_{\ell m}(v)\, P_\ell\left(\frac{v_n}{v}\right)\, dv$$

and

$$S_{\ell m}(v_n) = \int_{v_n}^{\infty} v\, h_{\ell m}(v)\, P_\ell\left(\frac{v_n}{v}\right)\, dv, \tag{116}$$

and for the first derivative we would get

$$\frac{dI}{dV_p} = 2\pi\frac{A_p e^2}{m}\sum_{\ell=0}^{\infty}\left[Q_\ell(v_n)P_\ell(\cos\theta_0) + \sum_{m=1}^{\ell} P_\ell^m(\cos\theta_j)\left\{R_{\ell m}(v_n)\cos m\phi_0 + S_{\ell m}(v_n)\sin m\phi_0\right\}\right]. \tag{117}$$

For calculating the net convective flow of the electrons $\langle\vec{v}\rangle$ it is convenient to show the $\ell = 0$ and $\ell = 1$ terms of the EVDF explicitly:

$$f(v,\theta,\phi) = f_0(v) + f_1(v)\cos\theta + g_{11}(v)\sin\theta\,\cos\phi + h_{11}(v)\sin\theta\,\sin\phi + \cdots\ . \tag{118}$$

Using the usual orthogonality ideas yields

$$\langle\vec{v}\rangle = \langle(v\sin\theta\cos\phi,\ v\sin\theta\sin\phi,\ v\cos\theta)\rangle$$
$$= \frac{4\pi}{3n_e}\left(\int_0^\infty v^3 g_{11}(v)dv,\ \int_0^\infty v^3 h_{11}(v)dv,\ \int_0^\infty v^3 f_1(v)dv\right). \tag{119}$$

B. Two-sided planar probe

Adding the contributions from both sides of the probe gives

$$\frac{d^2I}{dV_p^2} = 2\pi A_p \frac{e^3}{m^2} \times \sum_{\ell m} \frac{Y_{\ell m}(\theta_0,\phi_0) + Y_{\ell m}(\pi-\theta_0,\phi_0+\pi)}{2} F_{\ell m}(\epsilon^*), \quad (120)$$

and using the symmetry property

$$Y_{\ell m}(\pi-\theta_0,\phi+\pi) = (-1)^\ell Y_{\ell m}(\theta,\phi) \quad (121)$$

gives

$$\frac{d^2I}{dV_p^2}(\theta_0,\phi_0) = 2\pi A_p \frac{e^3}{m^2} \sum_{j=0}^{\infty} \sum_{m=-2j}^{+2j} Y_{2j,m}(\theta_0,\phi_0) \times F_{2j,m}(\epsilon^*). \quad (122)$$

We see that only even ℓ's contribute to the second derivative, but both even and odd m's do. If we calculate the electron density by integrating the second derivative according to the familiar algorithm for isotropic plasmas, we will obtain

$$n_e^{calc} = \frac{4\pi}{m}\sqrt{\frac{2}{m}} \sum_{j,m} Y_{2j,m}(\theta_0,\phi_0) \int_0^\infty F_{2j,m}(\epsilon^*)\epsilon^{*1/2}\, d\epsilon^*. \quad (123)$$

By exactly the same argument used in the axisymmetric case (Sec. III B) the integral in this equation vanishes unless $j = 0$. Using $F_{00} = f_{00}$, $Y_{00} = (4\pi)^{-1/2}$, and (from the orthogonality of the spherical harmonics)

$$f_{00}(\epsilon^*) = \int Y_{00}^* f(\epsilon^*,\theta,\phi)\, d\Omega \quad (124)$$

allows Eq. (123) to be simplified to

$$n_e^{calc} = \int_0^\infty v^2 \int f(v,\theta,\phi) d\Omega\, dv = \int_{\substack{velocity \\ space}} f = n_e^{true}. \quad (125)$$

This constitutes a second proof of our statement in Sec. II B that a two-sided planar probe always yields the correct density at any orientation in an arbitrarily anisotropic plasma.

C. Cylindrical probes

As in the case of a planar probe we choose three Cartesian coordinate systems in velocity space: (I) (X,Y,Z) fixed in the plasma vessel, (II) (x,y,z) as in Sec. III C with z axis parallel to probe axis and x pointing radially outward at a surface element on the sheath boundary, and (III) (x',y',z') with z' parallel to the electron velocity vector and x', y' chosen in any way. The rotations between these are as follows:

$$\begin{aligned} \bar R(\bar\phi,\bar\theta,\bar\chi) &: \text{(I)} \to \text{(III)}, \\ R(\phi,\theta,\chi) &: \text{(II)} \to \text{(III)}, \\ \tilde R(\Phi,\lambda,\tilde\phi) &: \text{(I)} \to \text{(II)}. \end{aligned} \quad (126)$$

The multiplication laws are

$$\bar R(\bar\phi,\bar\theta,\bar\chi) = \tilde R(\Phi,\lambda,\tilde\phi) R(\phi,\theta,\chi),$$
$$D^{(\ell)}_{mm'}(\bar\phi,\bar\theta,\bar\chi) = \sum_{m''} D^{(\ell)}_{mm''}(\Phi,\lambda,\tilde\phi) D^{(\ell)}_{m''m'}(\phi,\theta,\chi), \quad (127)$$

and specializing to $m' = 0$ and using Eq. (100) the latter becomes

$$Y_{\ell m}(\bar\theta,\bar\phi) = \sum_{m''} D^{(\ell)*}_{mm''}(\Phi,\lambda,\tilde\phi) Y_{\ell m''}(\theta,\phi). \quad (128)$$

More specifically (θ,ϕ) are the spherical coordinate angles of the electron velocity vector in the axis system at a surface element on the sheath boundary, $(\bar\theta,\bar\phi)$ are the spherical angles of the velocity vector in the vessel-fixed system used to express the EVDF, and (λ,Φ) are the spherical angles of the probe axis (and z axis) in that same system [Fig. 3(a)]. The angle $\tilde\phi$ is the angle around the probe axis to the surface element, measured from the projection of the plasma Z axis plus π radians [Fig. 3(b)]. The EVDF becomes

$$f(v,\bar\theta,\bar\phi) = \sum_{\ell m m''} f_{\ell m}(v) D^{(\ell)*}_{mm''}(\Phi,\lambda,\tilde\phi) Y_{\ell m''}(\theta,\phi), \quad (129)$$

and it is this expression that should be inserted into Eq. (76) in place of $f(v,\bar\theta)$ in order to represent the collected current. The $\tilde\phi$ integral is done first in the resulting expression. Using Eq. (100) we find

$$\int_0^{2\pi} d\tilde\phi \sum_{m''} D^{(\ell)*}_{mm''}(\Phi,\lambda,\tilde\phi) Y_{\ell m''}(\theta,\phi) = 2\pi Y_{\ell m}(\lambda,\Phi) P_\ell(\cos\theta). \quad (130)$$

Substituting (130) into the expression for collected current and doing the ϕ integral explicitly [as in going from Eq. (77) to Eq. (78)] we obtain a result that can be cast into the form of Eq. (80) if we define

$$f_{even}(v,\theta) \equiv \sum_{j=0}^{\infty} \sum_{m=-2j}^{+2j} f_{2j,m}(v) Y_{2j,m}(\lambda,\Phi) P_{2j}(\cos\theta). \quad (131)$$

The steps from Eq. (80) to Eq. (84) are exactly the same as before, and inserting the definition in Eq. (131) into Eq. (84) yields

$$\frac{d^2I}{dV_p^2} = 2\pi \frac{A_p e^3}{m^2} \sum_{jm} Y_{2j,m}(\lambda,\Phi) \left[f_{2j,m}(\epsilon^*) P_{2j}(0) + \int_{\epsilon^*}^\infty \frac{f_{2j,m}(\epsilon)}{\pi a^2 \epsilon} \int_0^a \frac{P'_{2j}(y) y\, dy}{\sqrt{a^2-y^2}}\, d\epsilon \right]. \quad (132)$$

Using Eq. (86) on this gives the final result

$$\frac{d^2I}{dV_p^2}(\lambda,\Phi) = 2\pi\frac{A_p e^3}{m^2}\sum_{j=0}^{\infty}\sum_{m=-2j}^{+2j} F_{2j,m}(\epsilon^*) \times Y_{2j,m}(\lambda,\Phi)P_{2j}(0), \tag{133}$$

where $F_{2j,m}(\epsilon^*)$ is obtained by putting $\ell = 2j$ into Eq. (107). This is very similar to the result for a two-sided planar probe [Eq. (122)], from which it differs only by the presence of the extra factor of $P_{2j}(0)$ and the substitution of λ,Φ for θ_0,ϕ_0. Since only even values of ℓ contribute to the second derivative, the same argument given in Sec. IV B shows that the electron density obtained by integrating the second derivative is the true one, providing a second proof of our assertion (from Sec. II C) that this method gives the exact result for any orientation of a cylindrical probe in an arbitrarily anisotropic plasma.

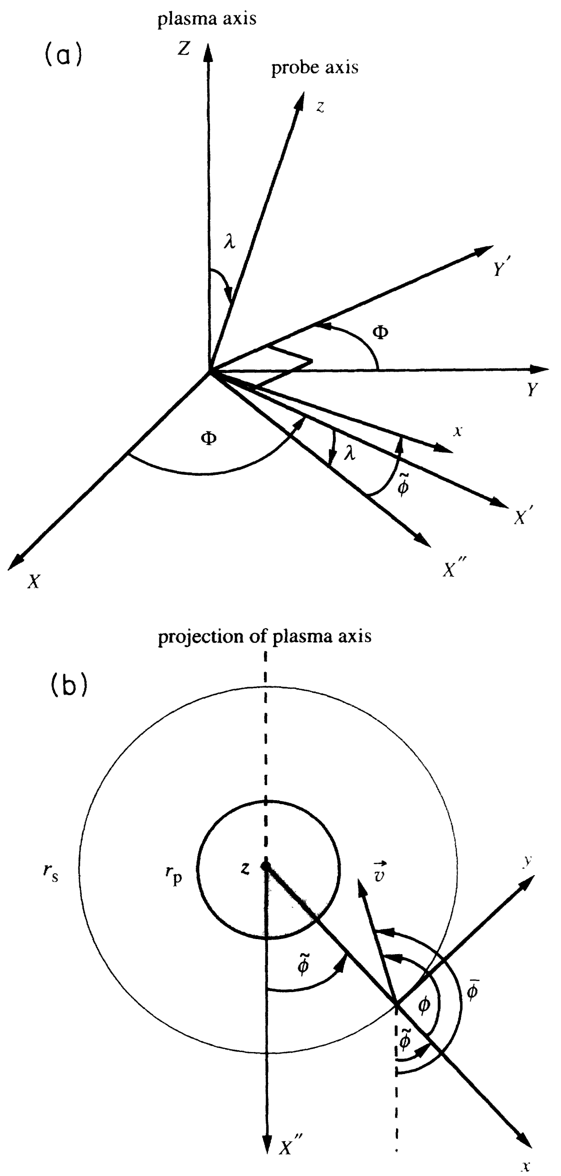

FIG. 3. (a) The Euler angles for the rotation $\tilde{R}(\Phi,\lambda,\tilde{\phi})$ are illustrated in perspective view. We first rotate by Φ around Z (plasma axis) to bring X to X' in the probe-axis–plasma-axis plane and Y' perpendicular to it. Then we rotate about Y' by λ to bring Z to z (the probe axis) and X' to X'', which is in the probe-axis–plasma-axis plane and perpendicular to the probe axis. Finally we rotate by $\tilde{\phi}$ around z to bring X'' to x, which points radially outward from the local surface element. (b) The view looking down the probe axis from the $+z$ direction.

For an axisymmetric plasma Eq. (133) becomes Eq. (93) as it should, again using the renormalization in Eq. (111). A version of Eq. (133) involving only real numbers can also be written. It will look like Eq. (115) with $\ell \to 2j, \theta_0,\phi_0 \to \lambda,\Phi$, and an extra factor of $P_{2j}(0)$ in each term. Using either that real equation or Eq. (133) itself, any chosen number of coefficients in the expansion of the EVDF (with even ℓ values) can be obtained from cylindrical probe I-V second derivative data at the same number of well chosen probe orientations by multiplying the column vector of I'' functions by the inverse of the coefficient matrix of the linear equations. Then the Volterra integral equation must be inverted by numerical integration.

D. Spherical probe

Let (v,θ,ϕ) be the spherical coordinates of the electron velocity vector in a system located at an infinitesimal surface element on the spherical sheath boundary, with the polar axis pointing radially outward from the probe center. Let (λ,Φ) be the spherical angles of that surface element in the vessel-fixed coordinate system and $(\bar{\theta},\bar{\phi})$ those of the velocity vector in the latter system. Then Eq. (129) still applies, with $\tilde{\phi}$ being arbitrary. The orbital motion is treated by conserving energy and angular momentum, leading to the following expression for the contribution of the surface element to the collected current:

$$dI = -dA\, e\int_0^{2\pi} d\phi \int_{\sqrt{\frac{-2eV_p}{m}}}^{\infty} v^2 dv \int_{\pi-\theta^*}^{\pi} \sin(\theta)d\theta \times v\cos(\theta) f(v,\bar{\theta},\bar{\phi}), \tag{134}$$

where

$$\theta^* = \sin^{-1}\frac{r_p}{r_s}\sqrt{1+\frac{2eV_p}{mv^2}}. \tag{135}$$

Combining Eq. (129) and (134) gives the expression for the total probe current:

$$I = -er_s^2\sum_{\ell m m''}\left[\int_0^{2\pi} d\Phi\int_0^{\pi}\sin(\lambda)d\lambda D^{(\ell)^*}_{mm''}(\Phi,\lambda,\tilde{\phi})\right] \times \int_{\sqrt{\frac{-2eV_p}{m}}}^{\infty} v^3 f_{\ell m}(v)\,dv \int_{\pi-\theta^*}^{\pi}\sin(\theta)\cos(\theta)\,d\theta \times\int_0^{2\pi} d\phi Y_{\ell m''}(\theta,\phi). \tag{136}$$

Using Eq. (104) with $\cos^{-1}\sqrt{\epsilon^*/\epsilon}$ replaced by θ and ϕ_1 by ϕ, followed by summing over m'' forces $m'' \to 0$, and using Eq. (100) we get a factor of

$$\int_0^{2\pi} d\Phi \int_0^{\pi} \sin(\lambda)d\lambda\, Y_{\ell m}(\lambda,\Phi) = \sqrt{4\pi}\,\delta_{\ell,0}\delta_{m,0}. \tag{137}$$

Doing the sums over ℓ and m gives

$$I = -\sqrt{4\pi}\, e r_s^2\, 2\pi \int_{\sqrt{\frac{-2eV_p}{m}}}^{\infty} v^3 f_{00}(v)\, dv \times \int_{\pi-\theta^*}^{\pi} \sin(\theta)\cos(\theta)d\theta, \tag{138}$$

after which the θ integral can be done explicitly, yielding

$$I = \pi e \frac{A_p}{\sqrt{4\pi}} \int_{\sqrt{\frac{-2eV_p}{m}}}^{\infty} v^3 f_{00}(v) \left(1 + \frac{2eV_p}{mv^2}\right) dv. \tag{139}$$

Differentiating twice and recalling that $f_{00} = \sqrt{4\pi} f_0$ gives

$$\frac{d^2I}{dV_p^2} = \frac{2\pi A_p e^3}{m^2} f_0 \left(\sqrt{\frac{-2eV_p}{m}}\right). \tag{140}$$

Thus the second derivative method gives only the isotropic term in the anisotropic EVDF, and of course in view of Eq. (70) integration of this will give the exact electron density.

V. DISCUSSION

While it is straightforward to design a probe assembly that permits rotation around a single axis, arranging for the probe orientation to cover θ_0, ϕ_0 or λ, Φ space in a balanced way presents a challenging mechanical design problem. Careful control and measurement of probe position and angle are clearly crucial to achieving accuracy in the inversion of the systems of linear equations and in obtaining the EVDF. Using more than one probe holder to make measurements at one position may solve the coverage problem, but must be done carefully to avoid introducing angle or position errors.

The numerical implementation of the method in the last section will no doubt be best handled by a general computer program. Depending on the expected or observed extent of anisotropy and the number of probe orientations that can be achieved, and also on the type of probe, there are a great many possibilities for how many terms and which terms one may wish to keep in the expansion of the EVDF. One should be able to enter a list of chosen ℓ, m and another of the sets θ_0, ϕ_0 or λ, Φ employed, with the computer using this information to set up and invert the coefficient matrix on an *ad hoc* basis. If the number of angle combinations exactly equals the number of terms desired in the expansion, then the linear equations are solved directly. A program, however, should also have the flexibility to use more orientations or multiple data sets at given orientations and to find the best solutions for the $F_{\ell m}$ functions (or similar equivalent ones) by using a linear least squares procedure. This would achieve improvements in accuracy, and the well known statistical methods could estimate errors and correlations and determine how suitable the given set of measurement angles is for the chosen list of ℓ, m pairs. It should be possible in principle to combine data from more than one probe type in such an analysis, but one would need to proceed cautiously in order to avoid introducing systematic errors, e.g., from different degrees of plasma perturbation by different probe holders. Numerical differentiation of probe trace data and numerical integration to solve the Volterra equations and to find electron densities would easily be incorporated into the same analysis program. Reference [11] contains pertinent discussion of possible numerical methods and possible errors. The most troublesome numerical problems come from the singularities in the Volterra integral equations at the plasma potential. Near the latter the errors in the coefficients of the EVDF increase. The axisymmetric case would be handled by the same program by just omitting the use of any $m \neq 0$ coefficients. In an anisotropic plasma the floating potential, which is defined to some extent by the surface used to measure it, is expected to depend strongly on the orientation of the probe. Experimentally variations of 5 V or more can be observed even when only a moderate level of anisotropy is present. The plasma potential, however, should be a scalar quantity, independent of the method used to measure it. Whether the zero crossing of the I-V second derivative occurs at exactly V_s in all probe orientations is another question. Experimentally we have found the zero crossing to be quite stable towards probe rotation when the anisotropy is not too great, but the theoretical situation at higher levels of anisotropy in the general case needs further investigation. Some discussion of this problem is found in Ref. [15].

Often it may be convenient or necessary to rotate the coordinate axes for expressing the EVDF. In the diverging downstream region of an electron cyclotron resonance (ECR) or other high density processing reactors with cylindrical spatial symmetry there will normally be symmetrically equivalent off-axis points (common radial and axial coordinates) with a completely anisotropic EVDF at each one. In measurements the same coordinate system would typically be used at all points, e.g., Z chosen parallel to the chamber axis and X axis pointing north. The $f_{\ell m}(v)$ functions at different azimuthal positions would then be very different, even though the EVDF's are symmetrically equivalent. We can easily rotate from one coordinate axis system (X, Y, Z) to another (X', Y', Z') with a transformation $R(\alpha, \beta, \gamma)$ that leads to analogs of Eqs. (101) or (128) for the spherical harmonic functions. For the coefficient functions then the transformation law is

$$f'_{\ell m'}(v) = \sum_m f_{\ell m}(v) D^{(\ell)*}_{mm'}(\alpha, \beta, \gamma), \tag{141}$$

where α, β, γ are the Euler angles of the coordinate axis rotation from (X, Y, Z) to (X', Y', Z').

We have recently described [20] a workstation based Langmuir probe system that obtains EVDF's by numerical differentiation of probe I-V curves. In the present context such a scheme has an advantage in comparison to analog differentiation methods in that one can choose to do a first derivative based analysis, or a second derivative based one, or both, after the data collection is complete. We have used this system to carry out a study of the accuracy of various probe analysis methods, including the integration of the EVDF, for on-axis measurements with an aligned cylindrical probe in dc discharges in nitrogen and in helium [21]. Section II of this paper shows that the anisotropy expected to exist, especially in the positive column of the discharge, will not introduce any errors into the plasma densities obtained by integrating the EVDF. A separate paper [22] presents a detailed mapping of plasma parameters and EVDF's obtained by the on-axis cylindrical probe in these same discharges. The distribution functions, obtained as if the plasma were isotropic, show complex patterns in the double layer region at the boundary between the Faraday dark space and the positive column and throughout the striated positive column. The analysis in Sec. II shows the EVDF's we found in those experiments are the functions $\bar{f}(\epsilon)$ and in fact describe only the distribution of velocities transverse to the discharge tube axis; knowing them is equivalent to knowing $G(v_\perp)$. Recent work in this group [23] has used a one-sided planar probe in several orientations to study the EVDF's at many spatial points in a multidipole magnetic confined plasma, excited at 13.6 MHz by an inductively coupled planar spiral antenna [multidipole rf induction (MRFI) plasma]. In the source region azimuthal rf electron currents are induced by the rf magnetic fields, so in general there is no reason to expect that the EVDF is axisymmetric, even though the vacuum chamber is very close to being perfectly cylindrically symmetrical.

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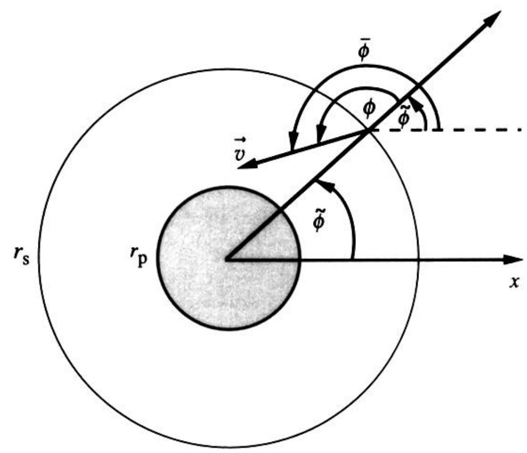

FIG. 1. The azimuthal angles are seen looking down the axis of the cylindrical probe, when the latter is also taken as the polar axis of the vessel-fixed cylindrical coordinate system for expressing the EVDF. The velocity vector is not generally in the plane of the paper; the angles shown are those to its projection in that plane.

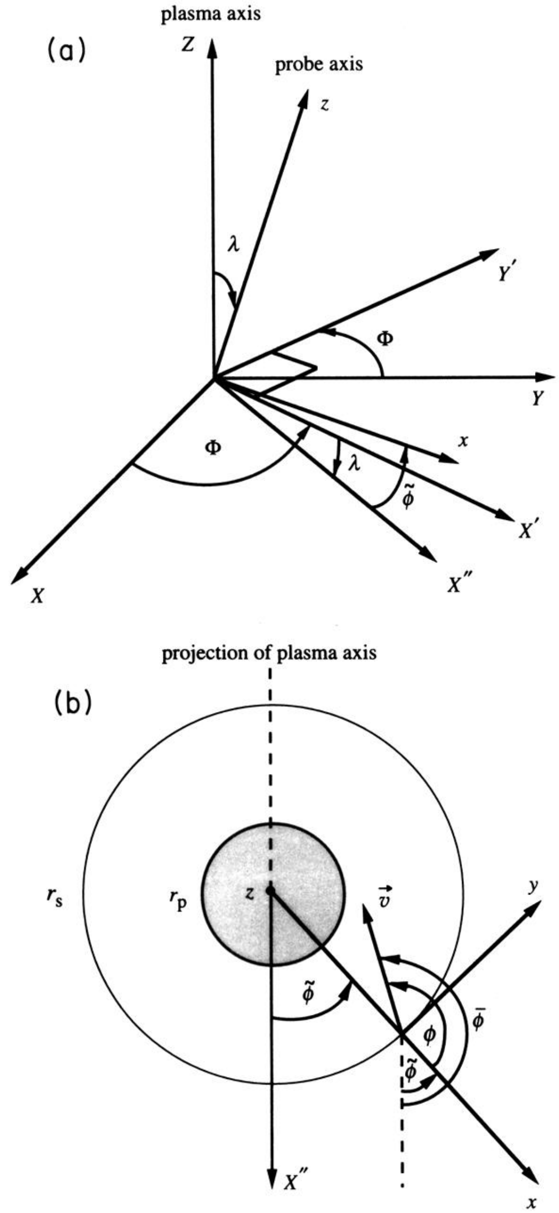

FIG. 3. (a) The Euler angles for the rotation $\tilde{R}(\Phi, \lambda, \tilde{\phi})$ are illustrated in perspective view. We first rotate by Φ around Z (plasma axis) to bring X to X' in the probe-axis–plasma-axis plane and Y' perpendicular to it. Then we rotate about Y' by λ to bring Z to z (the probe axis) and X' to X'', which is in the probe-axis–plasma-axis plane and perpendicular to the probe axis. Finally we rotate by $\tilde{\phi}$ around z to bring X'' to x, which points radially outward from the local surface element. (b) The view looking down the probe axis from the $+z$ direction.